Lecture 18

Wednesday, October 20, 2021 9:43 AM

Last Time: Fubinis Theorem

Ex. Compute $\iint_R y e^{-xy}\, dA$ on $R = [0,2] \times [0,3]$

Sol: $\iint_R y e^{-xy}\, dA = \underbrace{\int_{x=0}^{2} \underbrace{\int_{y=0}^{3} y e^{-xy}\, dy}_{\text{Compute inner integral}}\, dx}_{\text{Compute outer integral}}$

$$\int_0^3 y e^{-xy}\, dy$$

$u = y \qquad dv = e^{-xy} dy$

$du = du \qquad v = -\frac{1}{x} e^{-xy}$

$$-\frac{y}{x} e^{-xy} - \int -\frac{1}{x} e^{-xy}\, dy$$

$$-\frac{y}{x} e^{-xy} - \frac{1}{x^2} e^{-xy} \Big|_0^3$$

$$\left(-\frac{3}{x} e^{-3x} - \frac{1}{x^2} e^{-3x}\right) - \left(0 - \frac{1}{x^2}\right)$$

$$e^{-3x}\left(-\frac{3}{x} - \frac{1}{x^2}\right) + \frac{1}{x^2}$$

outer integral

$$\int_0^2 e^{-3x}\left(-\frac{3}{x} - \frac{1}{x^2}\right) + \frac{1}{x^2}\, dx$$

This does not work

Retry with y on outside x on inside

$$\underbrace{\int_{y=0}^{3} \underbrace{\int_{x=0}^{2} y e^{-xy}\, dx}_{\text{inner first}}\, dy}_{\text{outer last}}$$

$$\int_0^2 y e^{-xy}\, dx = -e^{-xy}\Big|_0^2 = -e^{-2y} - \left(-e^{-0y}\right)$$

$$= -e^{-2y} + 1$$

$$= -e^{-2y}+1$$

$$\int_0^3 -e^{-2y}+1 \; dy = y + \frac{1}{2}e^{-2y} \Big|_0^3$$

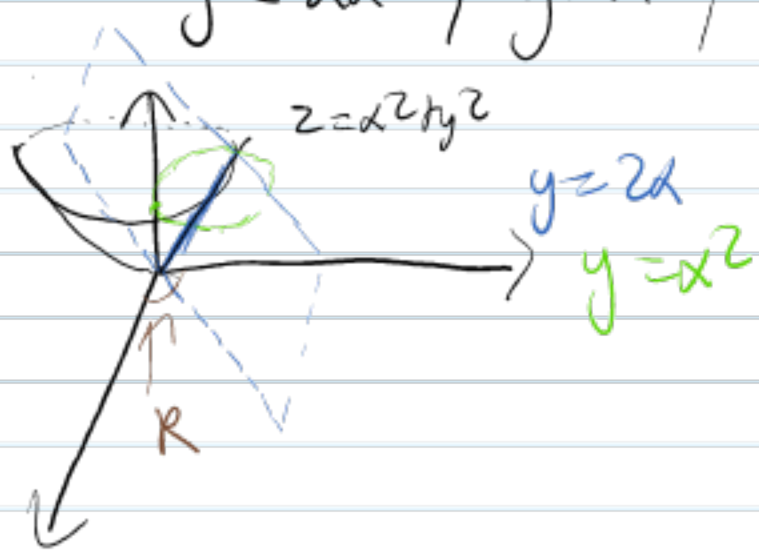
$$\left(3+\frac{1}{2}e^{-6}\right) - \left(0+\frac{1}{2}e^{0}\right) = \boxed{\frac{5}{2} + \frac{1}{2}e^{-6}}$$

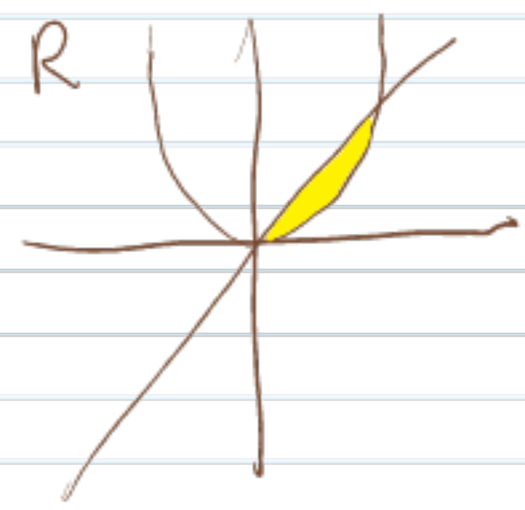
Goal: Integrate over a more complicated Region

Ex. Compute net volume of solid bound by

$$z = x^2+y^2 \qquad y = 2x, \quad y = x^2, \quad z = 0$$

Picture:





R $\{(x,y)$ (x,y) is between $y=2x$ and $y=x^2$

(x,y) $x^2 \le y \le 2x$

$$\text{Vol}(S) = \iint_R \left((x^2+y^2) - 0\right) dA$$

$$\int_{x=0}^{2} \int_{y=x^2}^{2x} (x^2+y^2)\, dy \, dx$$

$x^2 = 2x$

iff $x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0$, or $x = 2$

$$\int_{x^2}^{2x} x^2 + y^2 \; dy$$

$$yx^2 + \frac{y^3}{3} \Big|_{x^2}^{2x}$$

$$\int_{x=0}^{2} \left(2x^3 + \frac{(2x)^3}{3} - x^4 + \frac{(x^4)^3}{3}\right)$$

$$\int_{=0}^{2} \left(\frac{14}{3}x^3 - x^4 - \frac{1}{3}x^6\right) dx$$

$$\frac{14}{12}x^4 - \frac{1}{5}x^5 - \frac{1}{21}x^7 \Big|_0^2$$

$$\frac{14}{12}\cdot 2^4 - \frac{1}{5}\cdot 2^5 - \frac{1}{21}\cdot 2^7 - 0$$

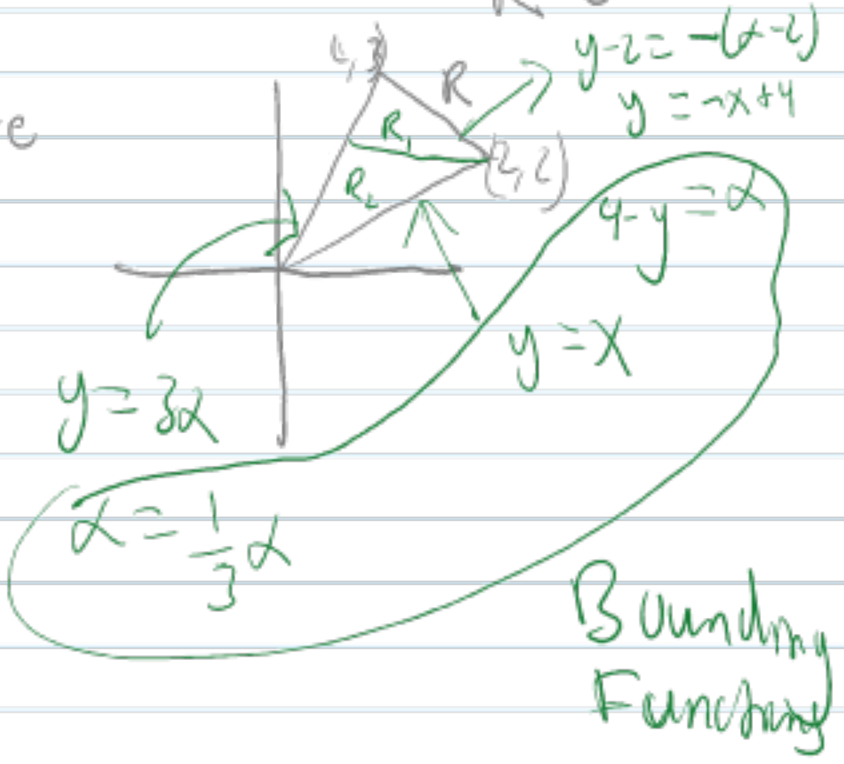
$$\frac{14}{12}\cdot 2^{4} - \frac{1}{5}\cdot 2^{5} - \frac{1}{21}\cdot 2^{7} - 0$$

$$\frac{14}{\cancel{12}\,3}\cdot\cancel{16}^{\,4} - \frac{1}{5}\cdot 32 - \frac{1}{21}\cdot 128$$

$$\boxed{\frac{56}{3} - \frac{32}{5} - \frac{128}{21}}$$

Ex. Compute $\iint_R y\,dA$ over R the triangle w/ vertices $(0,0),(1,3),(2,2)$

Picture



$R=\{(x,y) : 0\le y\le 3$

$R = R_1 \cup R_2$

$R_1 = \{(x,y) : 2\le y\le 3,\ \frac{1}{3}y \le x \le 4-y\}$

$R_2 = \{(x,y) : 0\le y\le 2,\ \frac{1}{3}y\le x\le y\}$

$$\iint_R y\,dA = \iint_{R_1} y\,dA + \iint_{R_2} y\,dA$$

$$\iint_{R_1} y\,dA = \int_{y=2}^{3}\int_{x=\frac{1}{3}y}^{4-y} y\,dx\,dy$$

$$\int_{\frac{1}{3}y}^{4-y} y\,dx$$

$$xy\Big/_{\frac{1}{3}y}^{4-y}$$

$$\int_{y=2}^{3}\left((4-y)y - \frac{1}{3}y\cdot y\right)dy$$

$$y\left(4-y-\frac{1}{3}y\right)$$

$$\iint_{R_2} y\,dA = \int_{y=0}^{2}\int_{x=\frac{1}{3}y}^{y} y\,dx\,dy$$

$$\int_{x=\frac{1}{3}y}^{y} y\,dx$$

$$xy\Big/_{\frac{1}{3}y}^{y}$$

$$\int_{y=0}^{2}\left(y^2-\frac{1}{3}y^2\right)dy$$

$$\int_{y=0}^{2} y^2\left(1-\frac{1}{3}\right)dy$$

$y(4-y - \frac{1}{3}y)$

$4y - y^2 - \frac{1}{3}y^2$

$\int_{y=2}^{3} \left(4y - \frac{4}{3}y^2\right) dy$

$2y^2 - \frac{4}{9} \Big|_2^3$

$R_1 = \frac{14}{9}$

$\int_{y=0} y\left(1 - \frac{y}{3}\right) dy$

$\frac{2}{3} \cdot \frac{1}{3} y^3 \Big|_0^2$

$\frac{2}{9}(8-0)$

$R_2 = \frac{16}{9}$

$$\iint_R y \, dA = \frac{14}{9} + \frac{16}{9}$$

$$= \boxed{\frac{10}{3}}$$

Motivating Question: What is the volume of a sphere?

Setup

$S = \{(x,y,z) : x^2+y^2+z^2 = r^2\}$

for $(x,y) \in R$

$z = \pm \sqrt{r^2 - x^2 - y^2}$

$\text{Vol}(S) = \iint_R 2\sqrt{r^2-x^2-y^2} \, dA$